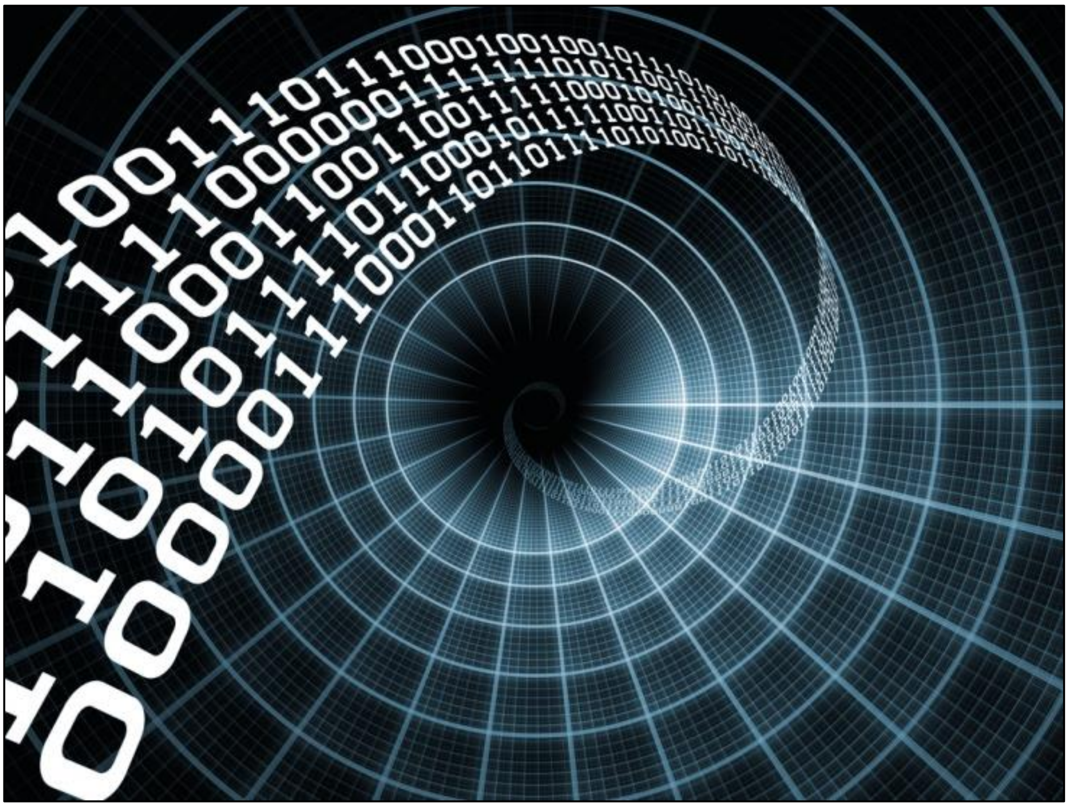
Course/Unit Support Notes

Advanced Higher Mathematics Course/Unit Support Notes



Please refer to the note of changes at the end of this document for details of changes from previous version (where applicable).

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Contents

Introduction

These support notes are not mandatory. They provide advice and guidance on approaches to delivering and assessing the Advanced Higher Mathematics Course. They are intended for teachers and lecturers who are delivering the Course and its Units.

These support notes cover both the Advanced Higher Course and the Units in it.

The Advanced Higher Course/Unit Support Notes should be read in conjunction with the relevant:

Mandatory Information:
- Course Specification
- Course Assessment Specification
- Unit Specifications

Assessment Support:
- Specimen and Exemplar Question Papers and Marking Instructions
- Exemplar Question Paper Guidance
- Guidance on the use of past paper questions
- Unit Assessment Support*

Related information

Advanced Higher Course Comparison

Further information on the Course/Units for Advanced Higher Mathematics

This information begins on page 11 and both teachers and learners may find it helpful.

General guidance on the Course/Units

Aims

The aims of the Course are to enable learners to:

- select and apply complex mathematical techniques in a variety of mathematical situations, both practical and abstract
- extend and apply skills in problem solving and logical thinking
- extend skills in interpreting, analysing, communicating and managing information in mathematical form, while exploring more advanced techniques
- clarify their thinking through the process of rigorous proof

Progression

In order to do this Course, learners should have achieved the Higher Mathematics Course.

Learners who have achieved this Advanced Higher Course may progress to further study, employment and/or training. Opportunities for progression include:

- Progression to other SQA qualifications
  - Progression to other qualifications at the same level of the Course, eg Mathematics of Mechanics, Statistics or Professional Development Awards (PDAs), or Higher National Certificates (HNCs)

- Progression to further/higher education
  - For many learners a key transition point will be to further or higher education, for example to Higher National Certificates (HNCs) or Higher National Diplomas (HNDs) or degree programmes.

  - Advanced Higher Courses provide good preparation for learners progressing to further and higher education as learners doing Advanced Higher Courses must be able to work with more independence and less supervision. This eases their transition to further/higher education. Advanced Higher Courses may also allow 'advanced standing' or partial credit towards the first year of study of a degree programme.

  - Advanced Higher Courses are challenging and testing qualifications — learners who have achieved multiple Advanced Higher Courses are regarded as having a proven level of ability which attests to their readiness for education in higher education institutions (HEIs) in other parts of the UK as well as in Scotland.

- Progression to employment
  - For many learners progression will be directly to employment or work-based training programmes.

This Advanced Higher is part of the Scottish Baccalaureate in Science. The Scottish Baccalaureates in Expressive Arts, Languages, Science and Social Sciences consist of coherent groups of subjects at Higher and Advanced Higher level. Each award consists of two Advanced Highers, one Higher and an Interdisciplinary Project, which adds breadth and value and helps learners to develop generic skills, attitudes and confidence that will help them make the transition into higher education or employment.

Hierarchies

Hierarchy is the term used to describe Courses and Units which form a structured progression involving two or more SCQF levels.

This Advanced Higher Course is not in a hierarchy with the corresponding Higher Course or its Units.

Skills, knowledge and understanding covered in this Course

This section provides further advice and guidance about skills, knowledge and understanding that could be included in the Course.

Teachers and lecturers should refer to the *Course Assessment Specification* for mandatory information about the skills, knowledge and understanding to be covered in this Course.

The development of subject-specific and generic skills is central to the Course. Learners should be made aware of the skills they are developing and of the transferability of them. It is the transferability that will help learners with further study and enhance their personal effectiveness.

The skills, knowledge and understanding that will be developed in the Advanced Higher Mathematics Course are:

- the ability to use mathematical reasoning skills to think logically, provide justification and solve problems
- knowledge and understanding of a range of complex concepts
- the ability to select and apply complex operational skills
- the ability to use reasoning skills to interpret information and to use complex mathematical models
- the ability to effectively communicate solutions in a variety of contexts
- the ability to explain and justify concepts through the idea of rigorous proof
- the ability to think creatively

Approaches to learning and teaching

Advanced Higher Courses place more demands on learners as there will be a higher proportion of independent study and less direct supervision. Some of the approaches to learning and teaching suggested for other levels (in particular, Higher) may also apply at Advanced Higher level but there will be a stronger emphasis on independent learning.

For Advanced Higher Courses, a significant amount of learning may be self-directed and require learners to demonstrate a more mature approach to learning and the ability to work on their own initiative. This can be very challenging for some learners, who may feel isolated at times, and teachers and lecturers should have strategies for addressing this. These could include, for example, planning time for regular feedback sessions/discussions on a one-to-one basis and on a group basis led by the teacher or lecturer (where appropriate).

Teachers and lecturers should encourage learners to use an enquiring, critical and problem-solving approach to their learning. Learners should also be given the opportunity to practise and develop research and investigation skills and higher order evaluation and analytical skills. The use of information and communications technology (ICT) can make a significant contribution to the development of these higher order skills as research and investigation activities become more sophisticated.

Learners will engage in a variety of learning activities as appropriate to the subject, for example:

- project-based tasks such as investigating the graphs of related functions, which could include using calculators or other technologies
- a mix of collaborative, co-operative or independent tasks which engage learners
- using materials available from service providers and authorities
- problem solving and critical thinking
- explaining thinking and presenting strategies and solutions to others
- effective use of questioning and discussion to engage learners in explaining their thinking and checking their understanding of fundamental concepts
- making links in themes which cut across the curriculum to encourage transferability of skills, knowledge and understanding — including with technology, geography, sciences, social subjects and health and wellbeing
- participating in informed debate and discussion with peers where they can demonstrate skills in constructing and sustaining lines of argument to provide challenge and enjoyment, breadth, and depth, to learning
- drawing conclusions from complex information
- using sophisticated written and/or oral communication and presentation skills to present information
- using appropriate technological resources (eg web-based resources)
- using appropriate media resources (eg video clips)

- using real-life contexts and experiences familiar and relevant to young people to meaningfully hone and exemplify skills, knowledge and understanding

Teachers and lecturers should support learners by having regular discussions with them and giving regular feedback. Some learning and teaching activities may be carried out on a group basis and, where this applies, learners could also receive feedback from their peers.

Teachers and lecturers should, where possible, provide opportunities to personalise learning and enable learners to have choices in approaches to learning and teaching. The flexibility in Advanced Higher Courses and the independence with which learners carry out the work lend themselves to this. Teachers and lecturers should also create opportunities for, and use, inclusive approaches to learning and teaching. This can be achieved by encouraging the use of a variety of learning and teaching strategies which suit the needs of all learners. Innovative and creative ways of using technology can also be valuable in creating inclusive learning and teaching approaches.

Centres are free to sequence the teaching of the Outcomes, Units and/or Course in any order they wish.

- Each Unit could be delivered separately in any sequence.

And/or:

- All Units may be delivered in a combined way as part of the Course. If this approach is used, the Outcomes within Units may either be partially or fully combined.

There may be opportunities to contextualise approaches to learning and teaching to Scottish contexts in this Course. This could be done through mini-projects or case studies.

Developing skills for learning, skills for life and skills for work

The following skills for learning, skills for life and skills for work should be developed in this Course.

2 Numeracy

2.1 Number processes
2.2 Money, time and measurement
2.3 Information handling

5 Thinking skills

5.3 Applying
5.4 Analysing and evaluating

Teachers and lecturers should ensure that learners have opportunities to develop these skills as an integral part of their learning experience.

It is important that learners are aware of the skills for learning, skills for life and skills for work that they are developing in the Course and the activities they are involved in that provide realistic opportunities to practise and/or improve them.

At Advanced Higher level, it is expected that learners will be using a range of higher order thinking skills. They will also develop skills in independent and autonomous learning.

Approaches to assessment

Assessment in Advanced Higher Courses will generally reflect the investigative nature of Courses at this level, together with high-level problem-solving and critical thinking skills and skills of analysis and synthesis.

This emphasis on higher order skills, together with the more independent learning approaches that learners will use, distinguishes the added value at Advanced Higher level from the added value at other levels.

There are different approaches to assessment, and teachers and lecturers should use their professional judgement, subject knowledge and experience, as well as understanding of their learners and their varying needs, to determine the most appropriate ones and, where necessary, to consider workable alternatives.

Assessments must be fit for purpose and should allow for consistent judgements to be made by all teachers and lecturers. They should also be conducted in a supervised manner to ensure that the evidence provided is valid and reliable.

Unit assessment

Units will be assessed on a pass/fail basis. All Units are internally assessed against the requirements shown in the *Unit Specification*. Each Unit can be assessed on an individual Outcome-by-Outcome basis or via the use of combined assessment for some or all Outcomes.

Assessments must ensure that the evidence generated demonstrates, at the least, the minimum level of competence for each Unit. Teachers and lecturers preparing assessment methods should be clear about what that evidence will look like.

Sources of evidence likely to be suitable for Advanced Higher Units could include:

- presentation of information to other groups and/or recorded oral evidence
- exemplification of concepts using (for example) a diagram
- interpretation of numerical data
- investigations
- case studies
- answers to (multiple choice) questions

Evidence should include the use of appropriate subject-specific terminology as well as the use of real-life examples where appropriate.

Flexibility in the method of assessment provides opportunities for learners to demonstrate attainment in a variety of ways and so reduce barriers to attainment.

The structure of an assessment used by a centre can take a variety of forms, for example:

- individual pieces of work could be collected in a folio as evidence for Outcomes and Assessment Standards
- assessment of each complete Outcome
- assessment that combines the Outcomes of one or more Units
- assessment that requires more than the minimum competence, which would allow learners to prepare for the Course assessment

Teachers and lecturers should note that learners' day-to-day work may produce evidence which satisfies assessment requirements of a Unit, or Units, either in full or partially. Such naturally-occurring evidence may be used as a contribution towards Unit assessment. However, such naturally-occurring evidence must still be recorded and evidence such as written reports, recording forms, PowerPoint slides, drawings/graphs, video footage or observational checklists, provided.

Combining assessment across Units

A combined approach to assessment will enrich the assessment process for the learner, avoid duplication of tasks and allow more emphasis on learning and teaching. Evidence could be drawn from a range of activities for a combined assessment. Care must be taken to ensure that combined assessments provide appropriate evidence for all the Outcomes that they claim to assess.

Combining assessment will also give centres more time to manage the assessment process more efficiently. When combining assessments across Units, teachers/lecturers should use e-assessment wherever possible. Learners can easily update portfolios, electronic or written diaries, and recording sheets.

For some Advanced Higher Courses, it may be that a strand of work which contributes to a Course assessment method is started when a Unit is being delivered and is completed in the Course assessment. In these cases, it is important that the evidence for the Unit assessment is clearly distinguishable from that required for the Course assessment.

Preparation for Course assessment

Each Course has additional time which may be used at the discretion of the teacher or lecturer to enable learners to prepare for Course assessment. This time may be used near the start of the Course and at various points throughout the Course for consolidation and support. It may also be used for preparation for Unit assessment, and, towards the end of the Course, for further integration, revision and preparation and/or gathering evidence for Course assessment.

For this Advanced Higher Course, the assessment method for Course assessment is a question paper. Learners should be given opportunities to practise this method and prepare for it.

Authenticity

In terms of authenticity, there are a number of techniques and strategies to ensure that learners present work that is their own. Teachers and lecturers should put in place mechanisms to authenticate learner evidence.

In Advanced Higher Courses, because learners will take greater responsibility for their own learning and work more independently, teachers and lecturers need to have measures in place to ensure that work produced is the learner's own work.

For example:

- regular checkpoint/progress meetings with learners
- short spot-check personal interviews
- checklists which record activity/progress
- photographs, films or audio records

Group work approaches are acceptable as part of the preparation for assessment and also for formal assessment. However, there must be clear evidence for each learner to show that the learner has met the evidence requirements.

For more information, please refer to SQA's *Guide to Assessment*.

Added value

Advanced Higher Courses include assessment of added value which is assessed in the Course assessment.

Information given in the *Course Specification* and the *Course Assessment Specification* about the assessment of added value is mandatory.

In Advanced Higher Courses, added value involves the assessment of higher order skills such as high-level and more sophisticated investigation and research skills, critical thinking skills and skills of analysis and synthesis. Learners may be required to analyse and reflect upon their assessment activity by commenting on it and/or drawing conclusions with commentary/justification. These skills contribute to the uniqueness of Advanced Higher Courses and to the overall higher level of performance expected at this level.

In this Course, added value will be assessed by means of a question paper. This is used to assess whether the learner can retain and consolidate the knowledge and skills gained in individual Units. It assesses knowledge and understanding and the various different applications of knowledge such as reasoning, analysing, evaluating and solving problems.

Equality and inclusion

It is recognised that centres have their own duties under equality and other legislation and policy initiatives. The guidance given in these *Course/Unit Support Notes* is designed to sit alongside these duties but is specific to the delivery and assessment of the Course.

It is important that centres are aware of and understand SQA's assessment arrangements for disabled learners, and those with additional support needs, when making requests for adjustments to published assessment arrangements. Centres will find more guidance on this in the series of publications on Assessment Arrangements on SQA's website: www.sqa.org.uk/sqa/14977.html.

The greater flexibility and choice in Advanced Higher Courses provide opportunities to meet a range of learners' needs and may remove the need for learners to have assessment arrangements. However, where a disabled learner needs reasonable adjustment/assessment arrangements to be made, you should refer to the guidance given in the above link.

Further information on Course/Units

The first column indicates the sub-skills associated with each Assessment Standard.

The second column is the mandatory skills, knowledge and understanding given in the *Course Assessment Specification*. This includes a description of the Unit standard and the added value for the Course assessment. Skills which could be sampled to confirm that learners meet the minimum competence of the Assessment Standards are indicated by a diamond bullet point (♦). Those skills marked by a diamond bullet point (♦) and those marked by an arrow bullet point (➢) can be assessed in the Course assessment.

For Unit assessment, when assessing sub-skills assessors should ensure that each ♦ associated with that sub-skill is assessed. Assessors can give learners access to the formulae contained in the formulae sheet accompanying the Advanced Higher Mathematics Course assessment. Assessors can also give learners access to any other derivative or formula which does not form part of this Course.

The third column gives suggested learning and teaching contexts to exemplify possible approaches to learning and teaching. These also provide examples of where the skills could be used in activities.

Lines and planes

In the Advanced Higher Mathematics Course the following definitions will apply.

Equation of a straight line in 3 dimensions

Vector form

The position vector, $\mathbf{r}$, of any point on the line is given by:

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \;\; (t \in \mathbb{R})$$

where $\mathbf{a}$ is the position vector of a point on the line and $\mathbf{b}$ is a vector in the direction of the line.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then the equation of the line can be written in the following forms where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Parametric form

$$x = a_1 + tb_1$$
$$y = a_2 + tb_2 \, , \; (t \in \mathbb{R})$$
$$z = a_3 + tb_3$$

Symmetric form

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} \;\; (= t)$$

Equation of a plane

The position vector, $\mathbf{r}$, of any point on the plane is given by:

Vector form

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c} \, , \; (s, t \in \mathbb{R})$$

where $\mathbf{a}$ is the position vector of a point on the plane and $\mathbf{b}$ and $\mathbf{c}$ are non-parallel vectors lying in the plane.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ then the equation of the plane can be written in the following forms where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Parametric form

$$x = a_1 + sb_1 + tc_1$$
$$y = a_2 + sb_2 + tc_2 \;\; (s, t \in \mathbb{R})$$
$$z = a_3 + sb_3 + tc_3$$

Cartesian or non-parametric form

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$$

where $\mathbf{a}$ is the position vector of a point on the plane and $\mathbf{n}$ is a vector normal to the plane.

This equation may also be written in the form

$$n_1 x + n_2 y + n_3 z = d$$

where $d = \mathbf{a.n}$ and $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$.

Mathematics: Methods in Algebra and Calculus (Advanced Higher)		
1.1 Applying algebraic skills to partial fractions		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Expressing rational functions as a sum of partial fractions (denominator of degree at most 3 and easily factorised)	♦ Express a proper rational function as a sum of partial fractions where the denominator may contain: distinct linear factors, an irreducible quadratic factor, a repeated linear factor: eg i) $\dfrac{7x+1}{x^2+x-6}$ (linear factors) ii) $\dfrac{5x^2-x+6}{x^3+3x}$ (irreducible quadratic factor) iii) $\dfrac{3x+10}{(x+3)^2}$ (repeated linear factor) iv) $\dfrac{7x^2-x+14}{(x-2)(x^2+4)}$ (linear factor and irreducible quadratic factor) ➢ Reduce an improper rational function to a polynomial and a proper rational function by division or otherwise eg $\dfrac{x^3+2x^2-2x+2}{(x-1)(x+3)}$ eg $\dfrac{x^2+3x}{x^2-4}$	This is required for integration of rational functions and useful in the context of differentiation and for graph sketching when asymptotes are present. This may be used with differential equations where the solution requires separating the variables. When the degree of the numerator of the rational function exceeds that of the denominator by 1, non-vertical asymptotes occur.

1.2 Applying calculus skills through techniques of differentiation		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Learners should be exposed to formal proofs of differentiation, eg from first principles, the product rule, quotient rule and chain although proofs will not be required for assessment purposes.		
Differentiating exponential and logarithmic functions	♦ Differentiate functions involving e^x, $\ln x$ eg $y = e^{3x}$ eg $f(x) = \ln\left(x^3 + 2\right)$	
Differentiating functions using the chain rule	♦ Apply the chain rule to differentiate the composition of at most 3 functions eg $y = \sqrt{e^{x^2} + 4}$ eg $f(x) = \sin^3(2x - 1)$	Learners should check answers and identify different ways of expressing their answers.

Differentiating functions given in the form of a product and in the form of a quotient	♦ Differentiate functions of the form $f(x)g(x)$ eg $y = 3x^4 \sin x$ eg $f(x) = x^2 \ln x,\ x > 0$ ♦ Differentiate functions of the form $\dfrac{f(x)}{g(x)}$ eg $y = \dfrac{2x-5}{3x^2+2}$ eg $f(x) = \dfrac{\cos x}{e^x}$ ♦ Know the definitions and use the derivatives of $\tan x$ and $\cot x$ ➢ Know the definitions of $\sec x,\ \operatorname{cosec} x$ — Learners should be able to derive and use derivatives of $\tan x,\ \cot x,\ \sec x,\ \operatorname{cosec} x$	
	➢ Differentiating functions which require more than one application or combination of applications of chain rule, product rule and quotient rule eg i) $y = e^{2x} \tan 3x$ ii) $y = \ln\lvert -3 + \sin 2x \rvert$ iii) $y = \dfrac{\sec 2x}{e^{3x}}$	Apply differentiation to rates of change, eg rectilinear motion and optimisation.

	iv) $y=\dfrac{\tan 2x}{1+3x^2}$ ➢ Know and use that $\dfrac{dy}{dx}=\dfrac{1}{\left(\dfrac{dx}{dy}\right)}$ ➢ Use logarithmic differentiation; recognise when it is appropriate in extended products, quotients, and in functions where the variable occurs in an index. eg $y=\dfrac{x^2\sqrt{7x-3}}{1+x}$, eg $y=2^x$, $y=x^{\tan x}$	
Differentiating inverse trigonometric functions	Differentiating expressions of the form ♦ $\sin^{-1} kx$ (Learners should know how to derive this.) ➢ $\tan^{-1}\left[f(x)\right]$ (Differentiate with the aid of the formulae sheet.)	Link with the graphs of these functions. Learners should be aware of $f^{-1}\left(f(x)\right)=x \Rightarrow \left(f^{-1}\right)'\left(f(x)\right)f'(x)=1 \Rightarrow \left(f^{-1}\right)'\left(f(x)\right)=\dfrac{1}{f'(x)}$
Finding the derivative of functions defined implicitly	♦ Use differentiation to find the first derivative of a function defined implicitly including in context. eg $x^3y+xy^3=4$ Apply differentiation to related rates in problems where the functional relationship is given implicitly, for example, the 'falling ladder' problem. ➢ Use differentiation to find the second derivative of a function defined implicitly.	Link with obtaining the derivatives of inverse trigonometric functions. Link with acceleration. $\dfrac{dv}{dt}=\dfrac{dv}{dx}\times\dfrac{dx}{dt}=v\dfrac{dv}{dx}$

Finding the derivative of functions defined parametrically	♦ Use differentiation to find the first derivative of a function defined parametrically including in context eg Apply parametric differentiation to motion in a plane If the position is given by $x = f(t)$, $y = g(t)$ then i) velocity components are given by $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$ ii) speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ eg Apply differentiation to related rates in problems where the functional relationship is given explicitly. $V = \frac{1}{3}\pi r^2 h$; given $\frac{dh}{dt}$, find $\frac{dV}{dt}$. ➢ Use differentiation to find the second derivative of a function defined parametrically ➢ Solve practical related rates by first establishing a functional relationship between appropriate variables	This could involve spherical balloons being inflated (or deflated) at a given rate, or calculating the rate at which the depth of coffee in a conical filter is changing.

1.3 Applying calculus skills through techniques of integration		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Learners will be expected to deal with definite or indefinite integrals as required.		
Integrating expressions using standard results	♦ Use $\int e^x\,dx,\ \int \frac{dx}{x},\ \int \sec^2 x\,dx$ eg $\int e^{5x-7}\,dx,\ \int \frac{dx}{2x-4},\ x \neq 2$	
	♦ Use the integrals of $\frac{1}{\sqrt{a^2-x^2}},\ \frac{1}{a^2+x^2}$	Link this with obtaining the derivatives of inverse trigonometric functions. Derivation of integrals on the formulae sheet should be included and may be assessed.
	➢ Recognise and integrate expressions of the form $\int g\left(f\left(x\right)\right) f'\left(x\right)dx$ and $\int \frac{f'\left(x\right)}{f\left(x\right)}dx$ eg $\int \cos^3 x \sin x\,dx$ eg $\int xe^{x^2}\,dx$ eg $\int_0^2 \frac{2x}{x^2+3}dx$ eg $\int \frac{\cos x}{\left(5+2\sin x\right)}dx$	Justification of these results could be used as examples of substitution on page 20.
	♦ Use partial fractions to integrate proper rational functions where the denominator may have:	Learning can be enhanced by completing the square.

	i) two separate or repeated linear factors ii) three linear factors with constant numerator eg $\int \frac{4x-9}{(x-2)(x-3)}dx$, $\int \frac{x+3}{(x+5)^2}dx$, $\int \frac{6}{(x-1)(x+2)(x+1)}dx$ ➢ Use partial fractions to integrate proper rational functions where the denominator may have: i) three linear factors with non-constant numerator ii) a linear factor and an irreducible quadratic factor of the form ax^2+bx+c	
Integrating by substitution	♦ Integrate where the substitution is given eg Use the substitution $u=\ln x$ to obtain $\int \frac{1}{x\ln x}dx,$ where $x>1.$	Learners may use substitution to integrate where a substitution has not been given provided it is a valid method. For example, an experienced learner could find, by inspection, $\int \frac{9x}{(3x^2+4)}dx$, however, it is possible to let $u=3x^2+4.$
Integrating by parts	♦ Use integration by parts with one application, eg $\int x\sin x dx$	Derive from the Product Rule. This may be revisited when using the integrating factor to solve first order differential equations. Consider cyclic integration by parts eg where $I_n = f(x)-2I_n$ eg $\int e^x \sin x\,dx$ eg $\int \ln x\,dx$ by considering $\ln x$ as $1.\ln x$

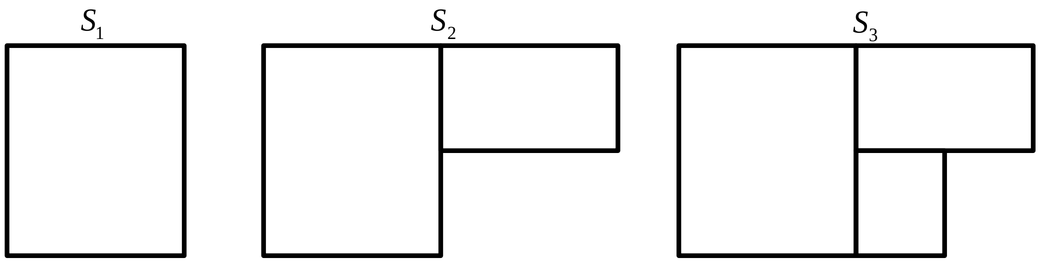
	➢ Use integration by parts involving repeated applications eg $\int_0^{\pi} x^2 \cos x \, dx$ eg $\int x^2 e^{3x} dx$	Applying integration to definite integrals.
1.4 Applying calculus skills to solving differential equations		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Solving first order differential equations with variables separable	♦ Solve equations that can be written in the form $\frac{dy}{dx} = g(x)h(y)$ or $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ eg $\frac{dy}{dx} = y(x-1)$ eg $v\frac{dv}{dx} = -\omega x$ ➢ Find general and particular solutions given suitable information eg $\frac{1}{x}\frac{dy}{dx} = y \sin x$ given that when $x = \frac{\pi}{2},\ y = 1$	Link with differentiation. Begin by verifying that a particular function satisfies a given differential equation. Learners should know that differential equations arise in modelling of physical situations, such as electrical circuits and vibrating systems, and that the differential equation describes how the system will change with time so that initial conditions are required to determine the complete solution. The most common use of this technique is for growth models these vary from (i) the rate of growth of population is proportional to either the size of the population [basic] or the space left to grow into [basic] (ii) the rate of growth of the population is proportional to both the population and the space left to grow into [extension] Scientific contexts such as chemical reactions, Newton's law of cooling, population growth and decay, bacterial growth and decay provide good motivating examples and can build on the knowledge and use of logarithms.

Solving first order linear differential equations using an integrating factor	♦ Solve equations written in the standard form $\frac{dy}{dx}+P(x)y=f(x)$ eg $\frac{dy}{dx}+\frac{3y}{x}=\frac{e^x}{x^3}$ ➢ Solve equations by first writing linear equations in the standard form $\frac{dy}{dx}+P(x)y=f(x)$ eg $x^2\frac{dy}{dx}+3xy=\sin x$	Contextualised problems where a generalised form of the differential equations is given.
	➢ Find general and particular solutions given suitable information.	This links to *Mathematical Techniques for Mechanics 1.4* and damped simple harmonic motion. Further examples of this could include: growth and decay problems, an alternative method of solution to separation of variables; simple electronic circuits. A mathematical example is illustrated below: A small tank of capacity 100 litres contains 10 kilograms of salt dissolved in 60 litres of water. A solution of water and salt (brine) with concentration of 0·1 kilograms per litre flows into the tank at the rate of 5 litres a minute. The solution in the tank is well-stirred and flows out at a rate of 3 litres per minute. How much salt is in the tank when the tank is full? The differential equation $\frac{dy}{dt}=\frac{-3}{2t+60}y+0\cdot5$ represents the amount of salt, y, present at time t. This can be solved subject to the condition $t=0,\ y=10$.

Solving second order differential equations	♦ Find the general solution and particular solution for initial value problems of second order homogeneous ordinary differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ where the roots of the auxiliary equation are real eg $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=0$ with $y=-1$ and $\frac{dy}{dx}=2$ when $x=0$ eg $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$ with $y=3$ and $\frac{dy}{dx}=1$ when $x=0$	Search for a trial solution using $y=e^{mx}$ and hence derive the auxiliary equation $am^2+bm+c=0$ Link with complex numbers Context applications could include the motion of a spring, both with and without a damping term.
	➢ Find the general solution and particular solution of second order homogeneous ordinary differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ where the roots of the auxiliary equation are real or complex conjugates eg $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+13y=0$ ➢ Solve second order non-homogeneous ordinary differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f\left(x\right)$ with constant coefficients, using the auxiliary equation and particular integral method	The general solution is the sum of the general solution of the corresponding homogeneous equation (complementary function) and a particular solution. For assessment purposes, only cases where the particular solution can easily be found by inspection will be required, with the right-hand side being a low order polynomial or a constant multiple of $\sin x$, $\cos x$, or e^{kx}.

Mathematics: Applications in Algebra and Calculus (Advanced Higher)		
1.1 Applying algebraic skills to the binomial theorem and to complex numbers		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Expand expressions using the binomial theorem	♦ Use the binomial theorem $\left(a+b\right)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$, for $r, n \in \mathbb{N}$ Expand an expression of the form $\left(ax+b\right)^n$ where $n \le 5$, $a, b \in \mathbb{Z}$ ➢ Expand an expression of the form $\left(ax^p + by^q\right)^n$, where $a, b \in \mathbb{Q}$; $p, q \in \mathbb{Z}$; $n \le 7$. eg Expand $\left(3x - \frac{1}{2x}\right)^6$ ➢ Using the general term for a binomial expansion, find a specific term in an expression eg Find the coefficient of x^7 in $\left(\frac{2}{x} + x\right)^{11}$; eg Find the term independent of x in the expansion of $\left(3x^2 - \frac{2}{x}\right)^9$	Learners should become familiar with factorial, Pascal's triangle, the binomial coefficient and the corresponding notation, ${}^nC_r = \binom{n}{r}$. Learners should know the results $\binom{n}{r} = \binom{n}{n-r}$ and $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ and be encouraged to prove them directly. Learners could be encouraged to expand, $\left(a+b\right)^n$, $n \in \mathbb{N}$, starting with $n = 2$ and increasing its value. Learners could then find connections between the coefficients of each term in each expansion and the binomial coefficients and Pascal's triangle. Learners could then make a conjecture about the expansion of $\left(a+b\right)^n$ and the general term of $\left(a+b\right)^n$. The binomial expansion $\left(p+q\right)^n$, $n \in \mathbb{N}$, where p and q are probabilities summing to 1, is used in Statistics, *Data Analysis and Modelling* Unit.

Performing algebraic operations on complex numbers	♦ Perform all of the operations of addition, subtraction, multiplication and division ➢ Find the square root eg $\sqrt{8-6i}$ ➢ Find the roots of a cubic or quartic with real coefficients when one complex root is given. ➢ Solve equations involving complex numbers eg solve $z+i=2\overline{z}+1$ eg solve $z^2=2\overline{z}$	Learners should be made aware that complex numbers (term given by Gauss (1831)) were first introduced by Italian mathematicians (eg Cardano, 1501–76) in the 16th century. They were a necessary tool to help find the roots of cubic equations. It took many years before ‘imaginary’ numbers, as Descartes (1637) called them, became accepted. It was Bombelli (1572) who introduced the symbol ‘i’ and learners should note: the basis of complex numbers is that $\sqrt{-1}=i$ and know the definition of the set of complex numbers can be written as, $\mathbb{C}=\{a+ib:\ a,\,b\in\mathbb{R}\}$. Learners should understand that numbers have real and imaginary parts. For addition and subtraction the real parts are gathered together likewise for the imaginary parts. Multiplication of complex numbers can be done by algebraic techniques eg $(3+2i)(4-i)$. Division of complex numbers can be done by using the complex conjugates in a similar way to rationalising the denominator using the conjugate surd.

1.2 Applying algebraic skills to sequences and series		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Finding the general term and summing arithmetic and geometric progressions	Apply the rules on sequences and series to find ♦ the nth term ♦ sum to n terms ♦ common difference of arithmetic sequences ♦ common ratio of geometric sequences ➢ sum to infinity of geometric series ➢ determine the condition for a geometric series to converge eg $1+2x+4x^2+8x^3+\cdots$ has a sum to infinity if and only if $\lvert x\rvert<\frac{1}{2}$	Learners should be encouraged to derive the formulae for finding the general term and summing arithmetic and geometric series. Learners should know the definition of a partial sum, ie the summation of a finite number of terms of a sequence, beginning with the first term. Partial sums can be useful for exploring convergence/divergence. For example, consider the geometric series $\sum_{r=0}^{\infty}\frac{1}{2^r}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ The partial sums $S_1=1$, $S_2=1+\frac{1}{2}=\frac{3}{2}$, $S_3=1+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}$, $S_4=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{15}{8}$ etc suggest — but do not prove — that the series $\sum_{r=0}^{\infty}\frac{1}{2^r}$ converges to a limit of 2 and can be summed using standard formula. The partial sums could be approached geometrically:
 		In this instance, each successive square/rectangle added on has area ½ of the preceding square/rectangle. Learners could then be encouraged to answer the following questions: *Does this series appear to be converging to a limit? If so, explain why.* *If $0<r<1$ will the series converge? Explain your answer.* *What happens if $r>1$?*

Using the Maclaurin expansion to find a stated number of terms of the power series for a simple function	♦ Use the Maclaurin expansion to find a power series for a simple non-standard function eg $\dfrac{1}{1+x^2}$ ➢ Use the Maclaurin expansion to find a power series eg $e^{\sin x}$, $e^x \cos 3x$	Learners could be introduced to the concept of the Maclaurin expansion by applying it to, for example, sin x. Learners should be made aware that the Maclaurin expansion gives approximations to simple functions and the use of graphic packages could be used to reinforce this connection. Learners should be familiar with the standard power series expansions of e^x, $\sin x$, $\cos x$ and $\ln(1 \pm x)$. For unit assessment power series should be derived and not quoted.
1.3 Applying algebraic skills to summation and mathematical proof		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Applying summation formulae	♦ Know and use sums of certain series and other straightforward results and combinations thereof (formulae which appear on the formulae sheet are 'use' only) 1. $\sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n} r + \sum_{r=1}^{n} b$ 2. $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$ (use only) 3. $\sum_{r=1}^{n} k = kn$ eg Find $\sum_{r=1}^{16} 4r+3$ ➢ 4. $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$ (use only)	Learners should be aware that $\sum_{r=1}^{n} k = k+k+k+...+k = kn$ Learners could be exposed to the derivation of 1–4 as a way of reinforcing their knowledge. The results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ can be used in several areas of Advanced Higher Statistics. A nice extension exercise for learners could be to use the results opposite to derive the alternative formula for standard deviation which the learners would have across earlier in their mathematics career.

	♦ 5. $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$ (use only) ➢ 6. $\sum_{r=k+1}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k} f(r)$ ➢ 7. $\sum_{r=1}^{n} f(r+1) - \sum_{r=1}^{n} f(r) = f(n+1) - f(1)$	
Using proof by induction	♦ use mathematical induction to prove summation formulae eg $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$ ➢ use proof by induction eg show that, $1+2+2^2+\cdots+2^n = 2^{n+1}-1,\ \forall n \in \mathbb{N}$ eg 8^n is a factor of $(4n)!,\ \forall n \in \mathbb{N}$ eg $y = x^n$, then $\frac{dy}{dx} = nx^{n-1},\ n \in \mathbb{N}$	Learners could be encouraged to prove some of the results and conjectures they have made throughout the Course by using mathematical induction. Learners should be made aware of the importance of a thorough understanding of the concept of induction.
1.4 Applying algebraic and calculus skills to properties of functions		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Finding the asymptotes to the graphs of rational functions	♦ Find the vertical asymptote to the graph of a rational function eg $f(x) = \frac{x^2+2x+4}{x-1}$ ♦ Find the non-vertical asymptote to the graph of a rational function	Learners could revisit the graph of $y = \tan x$ and explain what is happening as x tends towards $\frac{\pi}{2}$. This could aid their understanding of the concept of asymptotes. Learners could be encouraged to think of what happens to $\frac{1}{x}$ for very large x values and the impact this would have on functions.

	eg $f(x)=\dfrac{x^2+2x+4}{x-1}$, $f(x)=\dfrac{x^2}{x^2+1}$, $f(x)=\dfrac{x}{x^2+1}$	
Investigating features of graphs and sketching graphs of functions	♦ Investigate points of inflection, eg establish the coordinates of the point of inflection on the graph of $y=x^3+3x^2+2x$. ➢ Investigate other features: stationary points, domain and range, symmetry(odd/even), continuous/discontinuous, extrema of functions: the maximum and minimum values of a continuous function *f* defined on a closed interval $[a,b]$ can occur at stationary points, end points or points where f' is not defined eg calculate the maximum value, $0\le x\le 4$, of $f(x)=e^x\sin^2 x$ ➢ Sketch graphs using features given or obtained. ♦ Sketch related functions modulus functions inverse functions functions differentiated translations and reflections eg given $f(x)$ sketch the graph of $y=\lvert f(x)+a\rvert$ i) $f(x)=\sin 3x,\ \ y=\lvert 2f(x)-1\rvert$ ii) $f(x)=2x-7,\ \ y=\lvert 5-f(x)\rvert$	Learners should be able to determine the nature of stationary points by following the principles they used at Higher level. Learners should know that the second derivative test works because the second derivative gives the concavity at any point of a function. For example, for the function $f(x)=x^2$, $f''(x)=2$ which indicates that the function is concave up at all points and therefore any stationary point would be a minimum. They should also be aware that the second derivative test will not always work. Learners should be aware that points of inflexion occur where a function changes concavity.

1.5 Applying algebraic and calculus skills to problems		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Applying differentiation to problems in context	♦ Apply differentiation to problems in context eg A particle moves a distance s metres in t seconds. The distance travelled by the particle is given by $s = 2t^3 - \frac{23}{2}t^2 + 3t + 5$. Find the acceleration of the particle after 4 seconds. eg Apply differentiation to optimisation.	Students could be encouraged to consider from where the function $s(t)$ has been produced using IT to visualise the motion being modelled; to appreciate the gradient will be $v(t)$ and that similar IT considerations can be used to investigate its behaviour; to appreciate that the rate of change of v with time produces $a(t)$ and that finally, this modeling process should be interpreted in the context from which it arises. This is useful in least squares regression analysis in Statistics.
Applying integration to problems in context	♦ Apply integration to volumes of revolution where the volume generated is by the rotation of the area under a single curve about the x-axis ➢ Apply integration to volumes of revolution where the volume generated is by the rotation of the area under a single curve about the y-axis ➢ Use calculus to determine corresponding connected integrals ➢ Apply integration to the evaluation of areas including integration with respect to y	It might be useful to draw to the student's attention that the revolution can occur around the x-axis or around the y-axis and, in general, the results will be different. Initially avoid examples which may lead to special cases where the results are equal, eg $x^2 + y^2 = 1$ should not be your first choice. Other applications may include; given the velocity of a body, use integration to find displacement, Students should explore the geometric interpretation of the 'Fundamental Theorem of Calculus' to justify an interpretation of integrating with respect to y and why one would want to do it to make things easier.

Mathematics: Geometry, Proof and Systems of Equations (Advanced Higher)		
1.1 Applying algebraic skills to matrices and systems of equations		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Using Gaussian elimination to solve a 3 × 3 system of linear equations	♦ Find the solution to a system of equations $A\mathbf{x} = \mathbf{b}$, where A is a 3 × 3 matrix and where the solution is unique. Learners should understand the term augmented matrix. ➢ Show that a system of equations has no solutions (inconsistency) ➢ Show that a system of equations has an infinite number of solutions (redundancy) ➢ Compare the solutions of related systems of two equations in two unknowns and recognise ill-conditioning eg The systems of equations: $4x + 3y = 10$ $5x + 3\cdot 8y = 12\cdot 6$ has solution $x = 1,\ y = 2$ a slight change to the second equation giving: $5x + 3\cdot 7y = 12\cdot 6$ has solution $x = 4,\ y = -2$ the size of the change in the solution suggests the system is ill-conditioned.	Throughout this assessment standard, the use of CAS (computer algebra systems) may enhance learning, but when being assessed, learners will be expected to demonstrate all necessary skills. Use matrix ideas to organise a system of linear equations. Learners should be able to solve a 3 × 3 system of linear equations using Gaussian elimination on an augmented matrix. When solving a system of equations learners should use elementary row operations (EROs) to reduce the matrix to triangular form. This approach can also be used to explore situations where the system of equations is inconsistent or redundant. Links can be made with work on vectors to explore the geometric interpretation of the solution to a system of equations, eg the different ways 3 planes can intersect. The concept of ill-conditioning can be explored geometrically. Graphing calculators and/or computer software can support this investigative approach. Learners are not required to know any tests for ill-conditioning but should be able to recognise ill-conditioning through an analysis of the solutions to related systems.
Understanding and using matrix algebra	♦ Perform matrix operations (at most order 3) : addition, subtraction, multiplication by a scalar, multiplication of matrices ➢ Know and apply the properties of matrix addition and multiplication:	Learners should know the meaning of the terms: matrix, element, row, column, order, identity matrix, inverse, determinant, singular, non-singular, transpose. The condition for equality of matrices should be known. Learners should have the opportunity to explore associativity and distributivity as well as be exposed to proofs of these

	1. $A+B=B+A$ (addition is commutative) 2. $AB \neq BA$ (multiplication is not commutative in general) 3. $(A+B)+C=A+(B+C)$ (associativity) 4. $(AB)C=A(BC)$ (associativity) 5. $A(B+C)=AB+AC$ (addition is distributive over multiplication) ➢ Know and apply key properties of the transpose, the identity matrix, and inverse: 1. $\left(a_{ij}\right)'_{m\times n}=\left(a_{ji}\right)_{n\times m}$ ie rows and columns interchange 2. $(A')'=A$ 3. $(A+B)'=A'+B'$ 4. $(AB)'=B'A'$ 5. A matrix A is orthogonal if $A'A=I$ 6. The $n \times n$ identity matrix I_n: for any square matrix A , $AI_n=I_nA=A$ 7. $B=A^{-1}$ if $AB=BA=I$ 8. $(AB)^{-1}=B^{-1}A^{-1}$	properties. They should also explore and prove other key properties of the transpose, determinant and inverse (of a matrix). Learners should also be able to apply properties in combination to establish/derive other results. $I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $I_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ when clear from the contexts the subscripts can be omitted.
Calculating the determinant of a matrix	♦ Find the determinant of a 2 × 2 matrix and a 3 × 3 matrix. ♦ Determine whether a matrix is singular	The determinant of a 3 × 3 matrix can be calculated directly or first using EROs to introduce zeros. Learners could be encouraged to explore these and to establish their equivalence.

	➢ Know and apply $\det(AB) = \det A \det B$	Links can be made with the work on vectors. Learners should know that a matrix, A, is invertible $\Leftrightarrow \det A \neq 0$.
Finding the inverse of a matrix	♦ Know and use the inverse of a 2 × 2 matrix ➢ Find the inverse of a 3 × 3 matrix	When finding the inverse of a 3 × 3 matrix, links can be made with work on solving systems of equations. The role of the transpose in orthogonal cases should be considered and links established with the geometry of complex numbers. The inverse of a 3 × 3 matrix EROs can be found by using EROs or using the adjoint.
	➢ Use 2 × 2 matrices to carry out geometric transformations in the plane ➢ The transformations should include rotations, reflections and dilatations ➢ Apply combinations of transformations	Learners should explore and derive the various matrices associated with rotations, reflections and dilatations (enlargement/reduction). The role of the determinant and its geometric significance can also be investigated.
1.2 Applying algebraic and geometric skills to vectors		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Calculating a vector product	♦ Use a vector product method in three dimensions to find the vector product ➢ Evaluate the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$	Learners should already be familiar with the terms: position vector, unit vector, scalar product, components and the orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Learners should know the meaning of: vector product, scalar triple product and direction ratios/cosines. Learners should explore the link between the definition of $\mathbf{a} \times \mathbf{b}$ and the determinant of a 3 × 3 matrix. Links can be made with physics, eg finding the moment of a force about the origin, circular motion as well as with many other applications in engineering. The role of vectors in computer graphics/animation. The scalar triple product could be used to show learners how to calculate the volume of a parallelepiped.

Working with lines in 3 dimensions	♦ Find the equation of a line in parametric, symmetric or vector form, given suitable defining information. ➢ Find the angle between two lines in three dimensions ➢ Determine whether or not two lines intersect and, where possible, find the point of intersection	Learners could explore the equation of a line in 3D by analogy with 2D. The equation of a line in 2D provides sufficient information — starting at the origin — to get onto the line and then move along it. The 3D equivalent can be explored so that learners understand the requirement to know a point on the line and a direction vector. CAS provide a powerful tool to visualise this in 3D. Learners should be able to convert an equation from one form into another and should also be able to interpret the equations of lines given in any of the three forms. Learners should encounter situations where lines do not intersect.
Working with planes	♦ Find the equation of a plane in vector form, parametric form or Cartesian form given suitable defining information ➢ Find the point of intersection of a plane with a line which is not parallel to the plane ➢ Determine the intersection of 2 or 3 planes ➢ Find the angle between a line and a plane or between 2 planes	Learners should consider what information is necessary in order to uniquely locate a plane in 3D. Some questions to consider might be: Why are 2 points insufficient to uniquely define a plane? If we know 2 points on a plane what does this allow us to conclude? Why do 3 non-collinear points uniquely determine a plane? The ideas in this section are vital for further work in linear algebra. It would be desirable to provide learners with opportunities to explore the concept of linear independence: two linearly independent vectors in 3D will define a plane. The intersection of 3 planes should be linked to work on systems of equations to provide a geometric picture of redundancy and inconsistency. CAS can provide a powerful visual aid to justify the key skills and procedures being taught.

1.3 Applying geometric skills to complex numbers		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Performing geometric operations on complex numbers	♦ Plot complex numbers in the complex plane (an Argand diagram) ♦ Know the definition of modulus and argument of a complex number ♦ Convert a given complex number from Cartesian to polar form and vice-versa ➢ Use de Moivre's theorem with integer and fractional indices eg expand $(\cos\theta + i\sin\theta)^4$ ➢ Apply de Moivre's theorem to multiple angle trigonometric formulae eg express $\sin 5\theta$ in terms of $\sin\theta$ eg express $\sin^5\theta$ in terms of $\sin/\cos$ of multiples of θ ➢ Apply de Moivre's theorem to find the *n*th roots of a complex number eg solve $z^6 = 1$	Learners should know that $x = r\cos\theta, y = r\sin\theta$ where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$. Learners should also consider quadrants and ensure that for the principal value of the argument $-\pi < \theta \leq \pi$. Learners could be exposed to the exponential form of a complex number, $z = re^{i\theta}$. The Maclaurin series for $e^{\theta}, \cos\theta$ and $\sin\theta$ can be used to show that $e^{i\theta} = \cos\theta + i\sin\theta$ and hence that $e^{i\pi} + 1 = 0$. The proof of de Moivre's theorem for positive integers should be covered as an example of proof by induction. This could be extended to rational numbers and negative integers.
	➢ Interpret geometrically certain equations or inequalities in the complex plane ie find the loci defined by (in)equalities eg $\|z - i\| = \|z - 2\|$, $\|z - a\| > b$	Technology can be used to investigate loci.

1.4 Applying algebraic skills to number theory		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Using Euclid's algorithm to find the greatest common divisor of two positive integers	♦ Use Euclid's algorithm to find the greatest common divisor of two positive integers, ie use the division algorithm repeatedly ➢ Express the greatest common divisor (of two positive integers) as a linear combination of the two ➢ Express integers in bases other than ten ➢ Know and use the Fundamental Theorem of Arithmetic	Learners should know what is meant by: natural number, prime number, rational number, irrational number. For purposes of this Course: $\mathbb{N}=\{1,2,3,...\}$ $\mathbb{N}_0=\{0,1,2,3,...\}$ Learners should be exposed to a discussion of a proof of Euclid's algorithm. The general solution to the linear Diophantine equation $ax+by=c$ could be explored. As a further example, consider the 'postage stamp' problem: *What total sums of postage are possible using stamps whose values are a pence and b pence?* The use of hexadecimal (base 16) in computer programming is a possible context for teaching number bases. A proof of the Fundamental Theorem of Arithmetic should cover existence and uniqueness. Possible applications are used in modular arithmetic and cryptography.

1.5 Applying algebraic and geometric skills to methods of proof		
Sub-skill	**Description of Unit standard and added value**	**Learning and teaching contexts**
Disproving a conjecture by providing a counter-example	♦ Disprove a conjecture by providing a counter-example eg for all real values of a and b, $a-b>0 \Rightarrow a^2-b^2>0$. A counter-example is $a=3,\ b=-4$ ➢ Know and be able to use the symbols $\exists$ (there exists) and $\forall$ (for all)	Learners should know that a mathematical statement is a sentence which is either true or false (but not both). Use standard notation to form mathematical statements. Examples to focus on statements which may be: true for all real values of x; eg $\forall x \in \mathbb{R},\ x^2 \geq 0$

	➢ Write down the negation of a statement	true for at least one real value of x; eg $\exists\, y \in \mathbb{R} : y^2 = 4$ true for no real value of x. eg $x \in \mathbb{R} : x^2 = -1$ Learners should know what is meant by an existential statement, $(\exists)$ and a universal statement $(\forall)$. Learners should know the terms: negation $(\neg)$, conditional statement $(\Rightarrow)$, converse, contrapositive, statement, equivalence $(\Leftrightarrow)$. Learners should know and use the corresponding terminology: implies; implied by; if; only if; sufficient condition; necessary condition; if and only if (iff); necessary and sufficient condition. For direct proofs, learners should be aware that if $p \Rightarrow q$ and $q \Rightarrow r$ then $p \Rightarrow r$. The language of proof should be used correctly and other correct forms may be used, eg $x = 4$ is a sufficient condition for $x^2 = 16$ is equivalent to $x = 4$ only if $x^2 = 16$.

Using indirect or direct proof in straightforward examples	♦ Prove a statement by contradiction eg $\sqrt{2}$ is irrational eg if a and b are real then $a^2+b^2 \geq 2ab$ ➢ Use further proof by contradiction ➢ Use proof by contrapositive eg Prove that if n^2 is even then n is even eg If $x, y \in \mathbb{R}$: $x+y$ is irrational then at least one of x, y is irrational ♦ Use direct proof in straightforward examples eg Prove that the product of any 3 consecutive natural numbers is divisible by 6 eg Prove $m^2+n^2<(m+n)^2$, $\forall m, n \in \mathbb{N}$ or $m^2+n^2 \leq (m+n)^2$, $\forall m, n \in \mathbb{N}_0$	Teaching examples can focus on classic results, eg the infinitude of primes; the irrationality of $\sqrt{2}$. Learners could be encouraged to develop contrapositive statements from simple examples: 1. It is a polar bear implies it is a bear. 2. It is a not polar bear implies it is not a bear. 3. It is a bear implies it is a polar bear. 4. It is not a bear implies it is not a polar bear. It is useful for the student by this point to be familiar with the mathematical notation, ie the above could be written as: 1. $PB \Rightarrow B$ implication 2. $\neg PB \Rightarrow \neg B$ inverse (not true in this case) 3. $B \Rightarrow PB$ converse (not true in this case) 4. $\neg B \Rightarrow \neg PB$ contrapositive (has to be true if 1 is true) Learners should know that: $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$ the fact on which proof by contrapositive is based. Proofs using the contrapositive statement may be used in cases where a direct proof of $p \Rightarrow q$ may be more difficult, eg If n is a natural number such that n^2 is odd, then n is odd. That is $(n^2 \text{ odd}) \Rightarrow (n \text{ odd})$. By the contrapositive, $\neg(n \text{ odd}) \Rightarrow \neg(n^2 \text{ odd})$.

		Direct proof will feature prominently throughout the Course and should include the following: Standard results in differentiation from first principles; chain rule, product rule, quotient rule; other standard derivatives; integration by substitution; integration by parts; triangle inequality; sum of first n natural numbers; sum to n terms of arithmetic and geometric series; standard results in the algebra of vectors and matrices.

Appendix 1: Reference documents

The following reference documents will provide useful information and background.

- Assessment Arrangements (for disabled candidates and/or those with additional support needs) — various publications are available on SQA's website at: www.sqa.org.uk/sqa//14977.html.
- Building the Curriculum 4: Skills for Learning, Skills for Life and Skills for Work
- Building the Curriculum 5: A Framework for Assessment
- Course Specification
- Design Principles for National Courses
- *Guide to Assessment*
- Principles and practice papers for curriculum areas
- *SCQF Handbook: User Guide* and SCQF level descriptors
- *SQA Skills Framework: Skills for Learning, Skills for Life and Skills for Work*
- *Skills for Learning, Skills for Life and Skills for Work: Using the Curriculum Tool*
- *Coursework Authenticity: A Guide for Teachers and Lecturers*

Administrative information

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History of changes to Advanced Higher Course/Unit Support Notes

Version	Description of change	Authorised by	Date
2.0	Extensive changes to 'Further information on Course/Units' section.	Qualifications Development Manager	May 2015
2.1	'Further information on Course/Units' section: amendments to third sub-skill for Mathematics: Methods in Algebra and Calculus Unit, Assessment Standard 1.2 (Applying calculus skills through techniques of differentiation); amendments to second sub-skill for Mathematics: Applications in Algebra and Calculus Unit, Assessment Standard 1.5 (Applying algebraic and calculus skills to problems).	Qualifications Development Manager	December 2015
2.2	'Further information on Course/Units' section: Symbols for standard sets ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$) and for matrices standardised. Applications in Algebra and Calculus Assessment Standard 1.1: 'Finding the square root' changed to 'Find the square root' and 'Find the roots of a quartic with…' changed to 'Find the roots of a cubic or quartic with…'. Applications in Algebra and Calculus Assessment Standard 1.3: '2, 4, and 5 appear on formulae sheet and are therefore 'use' only' changed to 'formulae which appear on the formulae sheet are 'use' only'. Geometry, Proof and Systems of Equations Assessment Standard 1.3: 'ie find the loci defined by (in)equalities' added for clarification.	Qualifications Development Manager	May 2016

Note: You are advised to check SQA's website (**www.sqa.org.uk**) to ensure you are using the most up-to-date version.